8.4 Polar Representations for Complex Numbers

Imaginary Number: $i=\sqrt{-1}$ and $i^2=-1$

Complex Numbers: The set of all numbers of the form $a+bi$ where a and b are real numbers.

a is called the **real part** and b is called the **imaginary part**. If $b \neq 0,$ then $a+bi$ is an **imaginary number.** The form $a+bi$ is called the **standard form** of a complex number.

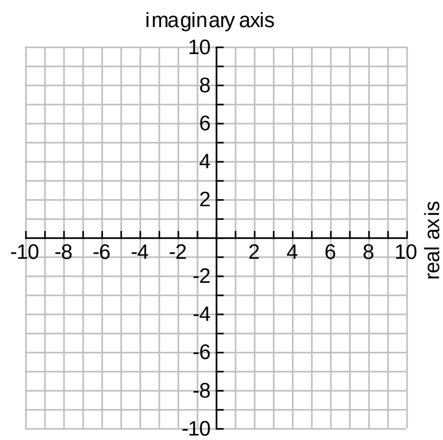
The complex number $a+bi$ can be thought of as an ordered pair (a,b). We graph it on the **complex plane** where the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis.**

Absolute Value or Modulus: $|a+bi|=\sqrt{a^2+b^2}$. (The distance between the number and the origin on the complex plane.)

Examples: Graph each complex number and find its absolute value.

a) $5-i$

b) $-6+2i$



The Modulus and Argument of Complex Numbers

Definition. The Modulus and Argument of Complex Numbers: Let $z=a+bi$ be a complex number with $a=\mathrm{Re}(z)$ and $b=\mathrm{Im}(z)$. Let (r,θ) be a polar representation of the point with rectangular coordinates (a,b) where $r \geq 0$.

- The **modulus** of z, denoted $|z|$, is defined by $|z|=r$.
- The angle θ is an **argument** of z. The set of all arguments of z is denoted $\arg(z)$.
- If $z \neq 0$ and $-\pi<\theta \leq \pi$, then θ is the **principal argument** of z, written $\theta=\mathrm{Arg}(z)$.

Example 8.4.1. For each of the following complex numbers find $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\arg(z)$ and $\mathrm{Arg}(z)$. Plot z in the complex plane.

1. $z = \sqrt{3} - i$

Solution.

1. For $z = \sqrt{3} - i = \sqrt{3} + (-1)i$, we have $\mathrm{Re}(z) = \sqrt{3}$ and $\mathrm{Im}(z) = -1$. To find $|z|$, $\arg(z)$ and $\mathrm{Arg}(z)$, we need to find a polar representation (r, θ) with $r \geq 0$ for the point $P\left(\sqrt{3}, -1\right)$ associated with z. We first determine a value for r.

$$r^2 = \left(\sqrt{3}\right)^2 + (-1)^2 \quad \textbf{from } r^2 = x^2 + y^2$$

$$r^2 = 4$$

$$r = \pm 2$$

We require $r \geq 0$, so we choose $r = 2$, and have $|z| = 2$.

Next, we find a corresponding angle θ. Since $r > 0$ and P lies in Quadrant IV, θ is a Quadrant IV angle. We have

$$\tan(\theta) = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \qquad \textbf{from } \tan(\theta) = \frac{y}{x}$$

$$\theta = -\frac{\pi}{6} + 2\pi k \text{ for integers } k \quad \textbf{since } \theta \textbf{ is a Quadrant IV angle}$$

Thus, $\arg(z) = \left\{ -\frac{\pi}{6} + 2\pi k \,\middle|\, k \text{ is an integer} \right\}$. Of these values, only $\theta = -\frac{\pi}{6}$ satisfies the requirement that $-\pi < \theta \leq \pi$, hence $\mathrm{Arg}(z) = -\frac{\pi}{6}$.

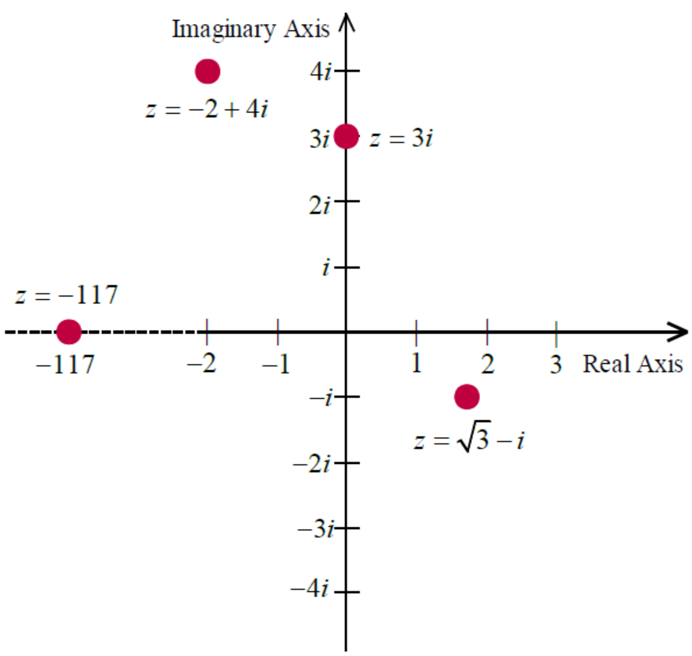
Now try:

2. $z = -2 + 4i$

3. $z = 3i$

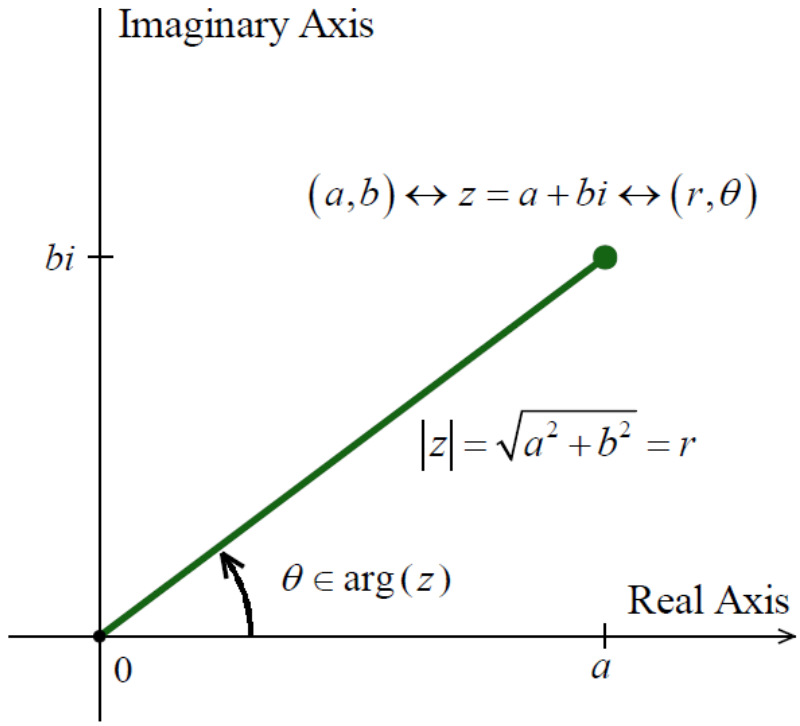
4. $z = -117$

We plot the four numbers from this example below.



Polar Form of Complex Numbers

Our next goal is to link the geometry and algebra of the complex numbers. To that end, consider the figure below.



Polar coordinate (r,θ) associated with $z = a+bi$, with $r \geq 0$

We know from **Theorem 8.1** that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Making these substitutions for a and b gives

$$\begin{aligned} z &= a+bi \\ &= r\cos(\theta) + r\sin(\theta)i \\ &= r\left[\cos(\theta) + i\sin(\theta)\right] \end{aligned}$$

The expression $\cos(\theta) + i\sin(\theta)$ is abbreviated $\operatorname{cis}(\theta)$ so we can write $z = r\operatorname{cis}(\theta)$. Since $r = |z|$ and $\theta \in \arg(z)$, we get

Definition. A Polar Form of a Complex Number: Suppose z is a complex number and $\theta \in \arg(z)$. The expression

$$|z|\operatorname{cis}(\theta) = |z|\left[\cos(\theta) + i\sin(\theta)\right]$$

is called a polar form for z.

Example 8.4.2. Find the rectangular form of the following complex numbers. Find $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$.

1. $z = 4\operatorname{cis}\left(\frac{2\pi}{3}\right)$

Solution. The key to this problem is to write out $\operatorname{cis}(\theta)$ as $\cos(\theta) + i\sin(\theta)$.

1. By definition,

$$\begin{aligned} z &= 4\operatorname{cis}\left(\frac{2\pi}{3}\right) \\ &= 4\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] \end{aligned}$$

After some simplifying, we get $z = -2 + 2i\sqrt{3}$, so that $\operatorname{Re}(z) = -2$ and $\operatorname{Im}(z) = 2\sqrt{3}$.

Now Try:

2. $z = 2\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

3. $z = 3\operatorname{cis}(0)$

4. $z = \text{cis}\left(\frac{\pi}{2}\right)$

Example 8.4.3. Use the results from **Example 8.4.1** to find a polar form of the following complex numbers.

1. $z = \sqrt{3} - i$

Solution. To write a polar form of a complex number z, we need two pieces of information: the modulus $|z|$ and an argument (not necessarily the principal argument) of z. We shamelessly mine our solution to **Example 8.4.1** to find what we need.

1. For $z = \sqrt{3} - i$, $|z| = 2$ and $\theta = -\frac{\pi}{6}$, so $z = 2\text{cis}\left(-\frac{\pi}{6}\right)$. We can check our answer by converting it back to rectangular form to see that it simplifies to $z = \sqrt{3} - i$.

Now try:

2. $z = -2 + 4i$

3. $z = 3i$

4. $z = -117$